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Symmetries and algebras of the integrable dispersive long wave equations in (2+1)-dimensional spaces

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Abstract. Similarly to the Kadomtsev–Petviashvili (KP) equation, a set of generalized symmetries with arbitrary functions of t is given by a simple constructable formula for the integrable dispersive long wave equations in 2+1 space dimensions. These symmetries constitute an infinite-dimensional Lie algebra which is a generalization to the known w_∞ algebra.

1. Introduction

The Kadomtsev–Petviashvili (KP) equation [1] (KP) is a (2+1)-dimensional system which leads to a large class of (1+1)-dimensional integrable models upon appropriate reductions [2, 3]. In terms of the dynamical variable $u(x, y, t)$, the KP equation can be written as

$$u_{xt} = (6uu_x - u_{xxx})_x - 3u_{yy} \equiv K_{2x}\,. \tag{1}$$

In [4], the generalized Lie algebra constituted by the symmetries of the KP equation has been obtained:

$$[K_n(h_1),\ K_m(h_2)] = \tfrac{1}{3}K_{n+m-2}((m+1)\dot{h}_1 h_2 - (n+1)\dot{h}_2 h_1) \tag{2}$$

where h_1 and h_2 are arbitrary functions of t, $\dot{h} = (\partial/\partial t)h = D_t h$ and the Lie product [,] is defined by

$$[A,\ B] = \frac{\partial}{\partial\epsilon}[A(u+\epsilon B) - B(u+\epsilon A)]\,|_{\epsilon=0} = A'B - B'A\,. \tag{3}$$

The generalized symmetries $K_n(h)$ can be expressed simply by [4]

$$K_n(h) = \frac{1}{2n!3^{n+1}}\sum_{k=0}^{n+1} h^{(n+1-k)}(K' - D_t)^k y^n \qquad n \geqslant 0 \tag{4}$$

with

$$K' = 6D_x u - D_x^3 - 3D_x^{-1}D_y^2 \qquad D_x = \frac{\partial}{\partial x} \quad D_y = \frac{\partial}{\partial y}\,.$$

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The Abelian algebra (h = constant), centreless Virasoro algebra ($h = t$), w_∞ algebra ($h = t^s,\ s \geqslant 0$) [5, 6] and the loop algebra obtained in [7] ($h_m = 0,\ m \geqslant 3$) are all special cases of the algebra (2) [4]. Now the question is whether a similar algebra can be found for other types of the integrable models.

In this paper, we study the generalized symmetries and algebras of the (2+1)-dimensional integrable model, integrable dispersive long wave equations (IDLWE)

$$u_{ty} = -\eta_{xx} - \tfrac{1}{2}(u^2)_{xy} \tag{5}$$

$$\eta_t = -(u\eta + u_{xy})_x \tag{6}$$

which were first obtained by Boiti *et al* [8]. Paquin and Winternitz had given a Kac–Moody–Virasoro symmetry algebra [9]. In [10], nine types of two-dimensional partial differential equation reductions and 13 types of the ordinary differential equation reductions are obtained by means of the direct and non-classical method [11, 3]. The IDLWE (5) and (6) have no Painlevé property [12] though they are integrable [8].

In section 2 of this paper we derive a set of generalized symmetries with arbitrary functions of t. This set of symmetries constitutes an infinite-dimensional Lie algebra. The commuting algebra, centreless Virasoro algebra and a w_∞ algebra are all some special subalgebras. Section 3 is devoted to discussing the existence of another set of symmetries with arbitrary functions of space y. Only one special symmetry of this type is given. Section 4 is a summary and discussion.

2. Symmetries with arbitrary functions of t and the corresponding algebra

A symmetry

$$\sigma = \begin{pmatrix} U \\ H \end{pmatrix} \tag{7}$$

of the IDLWE is defined as if σ satisfies the linearized equations of (5) and (6)

$$U_{ty} = -H_{xx} - (uU)_{xy} \tag{8}$$

$$H_t = -(U\eta + uH + U_{xy})_x \tag{9}$$

which means that (5) and (6) are form invariant under the transformation

$$u \longrightarrow u + \epsilon U \tag{10}$$

$$\eta \longrightarrow \eta + \epsilon H \qquad \epsilon \text{ infinitesimal.} \tag{11}$$

Similarly to the KP equation, we look for the symmetries of the IDLWE which have the form

$$U(f) = \sum_{k=0}^{n+1} f^{(n+1-k)} U[k] \tag{12}$$

$$H(f) = \sum_{k=0}^{m} f^{(m-k)} H[k] \tag{13}$$

with f being an arbitrary function of t, $f^{(k)} = (\partial^k/\partial t^k)f$ and $U[k]$ and $H[k-1]$ are functions of x, y, u, η and their derivatives, but they are not time-dependent explicitly. the explicit time dependencies of $U(f)$ and $H(f)$ have been separated out in $f(t)$ and its derivatives.

Substituting (12) and (13) into (8) and (9) and considering the non-trivial condition $\sigma \neq 0$ yields the only possibility:

$$m = n \tag{14}$$

$$\sum_{k=0}^{n+1} f^{(n+2-k)} U_y[k] + \sum_{k=1}^{n+2} f^{(n+2-k)} U_{ty}[k-1] = -\sum_{k=2}^{n+2} f^{(n+2-k)} H_{xx}[k-2] - \sum_{k=1}^{n+2} f^{(n+2-k)} D_x D_y u U[k-1] \tag{15}$$

$$\sum_{k=0}^{n} f^{(n+1-k)} H[k] + \sum_{k=1}^{n+1} f^{(n+1-k)} H_t[k-1] = -\sum_{k=1}^{n+1} f^{(n+1-k)} (uH[k-1])_x - \sum_{k=0}^{n+1} f^{(n+1-k)} \left(\eta U[k] + U_{xy}[k]\right)_x . \tag{16}$$

Since $f = f(t)$ is an *arbitrary* function of t, (15) and (16) should be true at any order of differentiation of $f(t)$. That means the following overdetermined equations should be satisfied:

$$U_y[0] = 0 \qquad \text{i.e.} \quad U[0] = g(x) = g \tag{17}$$

$$H[0] = -(\eta U[0])_x \tag{18}$$

$$U[1] = -(uU[0])_x \tag{19}$$

$$U[k] = -U_t[k-1] - D_y^{-1} H_{xx}[k-2] - (uU[k-1])_x \qquad k = 2, 3, \ldots, n+1 \tag{20}$$

$$H[k-1] = -H_t[k-2] - (uH[k-2])_x - (\eta U[k-1])_x - U_{xxy}[k-1] \qquad k = 2, 3, \ldots, n+1 \tag{21}$$

$$U_{ty}[n+1] = -H_{xx}[n] - (uU[n+1])_{xy} \tag{22}$$

$$H_t[n] = -(uH[n])_x - (\eta U[n+1])_x - U_{xxy}[n+1] \tag{23}$$

where $D_y D_y^{-1} = D_y^{-1} D_y = 1$. Equations (20) and (21) can be solved recursively; the result reads

$$\begin{pmatrix} U[k] \\ H[k-1] \end{pmatrix} = \begin{pmatrix} -D_t - D_x u & -D_y^{-1} D_x^2 \\ -D_x \eta - D_x^2 D_y & -D_t - D_x u \end{pmatrix} \begin{pmatrix} U[k-1] \\ H[k-2] \end{pmatrix} = \begin{pmatrix} -D_t - D_x u & -D_y^{-1} D_x^2 \\ -D_x \eta - D_x^2 D_y & -D_t - D_x u \end{pmatrix}^{k-1} \begin{pmatrix} -(ug)_x \\ -(\eta g)_x \end{pmatrix} \qquad k = 1, 2, \ldots, n+1 . \tag{24}$$

Now the only thing left to do is to substitute

$$\begin{pmatrix} U[n+1] \\ H[n] \end{pmatrix} = \begin{pmatrix} -D_t - D_x u & -D_y^{-1} D_x^2 \\ -D_x \eta - D_x^2 D_y & -D_t - D_x u \end{pmatrix}^{n} \begin{pmatrix} -(ug)_x \\ -(\eta g)_x \end{pmatrix} \qquad n \geqslant 1 \tag{25}$$

into (22) and (23) to determine the only unknown function $g(x)$.

Fortunately, as in the KP case, we can firstly give the form of $g(x)$ in a simple alternative way. At first, substituting $g(x) = 1,\ x/2$ and $x^2/8$ into (17)–(23) respectively, we have

$$U_0[0] = 1 \qquad H_0[0] = -\eta_x \qquad U_0[1] = -u_x \tag{28}$$

$$U_0[n] = H_0[n-1] = 0 \qquad n > 1 \tag{29}$$

$$U_1[0] = \tfrac{1}{2}x \qquad H_1[0] = -\tfrac{1}{2}(x\eta)_x \qquad U_1[1] = -\tfrac{1}{2}(xu)_x \tag{30}$$

$$H_1[1] = -\eta_t \qquad U_1[2] = -u_t \tag{31}$$

$$U_1[n] = H_1[n-1] = 0 \qquad n > 2 \tag{32}$$

$$U_2[0] = \tfrac{1}{8}x^2 \qquad H_2[0] = -\tfrac{1}{8}(x^2\eta)_x \qquad U_2[1] = -\tfrac{1}{8}(x^2u)_x \tag{33}$$

$$H_2[1] = \tfrac{1}{4}[2xu\eta + (2xu_x + u)_y]_x \tag{34}$$

$$U_2[2] = \tfrac{1}{4}[xu^2 + D_y^{-1}(2x\eta_x + \eta)]_x \tag{35}$$

$$H_2[2] = -\tfrac{1}{4}[3(uu_x)_y + 3u^2\eta + 3uu_{xy} + 3\eta D_y^{-1}\eta_x + 4\eta_{xx}]_x \tag{36}$$

$$U_2[3] = -\tfrac{1}{4}[3D_y^{-1}(u\eta)_x + u^3 + 3uD_y^{-1}\eta_x + 4u_{xx}]_x \tag{37}$$

$$U_2[n] = H_2[n-1] = 0 \qquad n > 3\,. \tag{38}$$

Comparing (29) and (23) with (8) and (9) we know that

$$\begin{pmatrix} -U[n+1] \\ -H[n] \end{pmatrix} \equiv K_n = K_n(-1)$$

itself is an f- (and then t-) independent symmetry of the IDLWE which is the same as saying that

$$\begin{pmatrix} U[n+2] \\ H[n+1] \end{pmatrix} = 0\,.$$

Consequently, (29), (31) and (36) tell us that

$$K_0(-1) \equiv \begin{pmatrix} -U_0[1] \\ -H_0[0] \end{pmatrix} = \begin{pmatrix} u_x \\ \eta_x \end{pmatrix} \tag{39}$$

$$K_1(-1) \equiv \begin{pmatrix} -U_1[2] \\ -H_1[1] \end{pmatrix} = \begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \tag{40}$$

$$K_2(-1) \equiv \begin{pmatrix} -U_2[3] \\ -H_2[2] \end{pmatrix} = \frac{3}{4}D_x \begin{pmatrix} (uu_x)_y + u^2\eta + uu_{xy} + \eta D_y^{-1}\eta_x + \frac{4}{3}\eta_{xx} \\ D_y^{-1}(u\eta)_x + uD_y^{-1}\eta_x + \frac{1}{3}u^3 + \frac{4}{3}u_{xx} \end{pmatrix} \tag{41}$$

are three time-independent symmetries of the IDLWE. The symmetries $K_0(-1)$ and $K_1(-1)$ correspond to the x and t translations respectively. Substituting (28)–(38) into (7) with (12) and (13), we get three generalizations of (39)–(41):

$$K_0(f) = \begin{pmatrix} -f(t)u_x + \dot{f}(t)x \\ -f(t)\eta_x \end{pmatrix} \tag{42}$$

$$K_1(f) = \begin{pmatrix} -f(t)u_t - \frac{1}{2}\dot{f}(xu)_x + \frac{1}{2}\ddot{f}x \\ -f(t)\eta_t - \frac{1}{2}\dot{f}(x\eta)_x \end{pmatrix} \tag{43}$$

$$K_2(f) = \begin{pmatrix} -\frac{3}{4}f[(uu_x)_y + u^2\eta + uu_{xy} + \eta D_y^{-1}\eta_x + \frac{4}{3}\eta_{xx}]_x \\ \quad + \frac{1}{4}\dot{f}[D_y^{-1}(2x\eta_x + \eta) + xu^2]_x - \frac{1}{8}\ddot{f}(x^2u)_x + \frac{1}{8}\dddot{f}x^2 \\ -\frac{3}{4}f[(D_y^{-1}(u\eta)_x + \frac{1}{3}u^3 + uD_y^{-1}\eta_x + \frac{4}{3}u_{xx}]_x \\ \quad + \frac{1}{4}\dot{f}[2xu\eta + (2xu_x + u)_y]_x - \frac{1}{8}\ddot{f}(x^2\eta)_x \end{pmatrix}. \tag{44}$$

If we take $f = t$ in $K_2(f)$ we get

$$K_2(t) = \begin{pmatrix} -\frac{3}{4}t[(uu_x)_y + u^2\eta + uu_{xy} + \eta D_y^{-1}\eta_x + \frac{4}{3}\eta_{xx}]_x \\ -\frac{3}{4}t[(D_y^{-1}(u\eta))_x + \frac{1}{3}u^3 + uD_y^{-1}\eta_x + \frac{4}{3}u_{xx}]_x \end{pmatrix} + \begin{pmatrix} \frac{1}{4}[D_y^{-1}(2x\eta_x + \eta) + xu^2]_x \\ \frac{1}{4}[2xu\eta + (2xu_x + u)_y]_x \end{pmatrix} \equiv -tK_2(-1) + M \,. \tag{45}$$

One can readily verify that

$$M = \begin{pmatrix} \frac{1}{4}[D_y^{-1}(2x\eta_x + \eta) + xu^2]_x \\ \frac{1}{4}[2xu\eta + (2xu_x + u)_y]_x \end{pmatrix} \tag{46}$$

is a master symmetry of degree one for the IDLWE. Now using the standard master symmetry approach [13, 14] and seed $K_0(-1)$, we get a set of the time-independent commuting symmetries:

$$K_n = K_n(-1) = -\frac{2}{n}[M,\ K_{n-1}] = -\frac{2}{n}[K_2(t),\ K_{n-1}] \qquad n = 1, 2, \ldots, K_{-1} = 0 \tag{47}$$

$$[K_m,\ K_n] = 0\,. \tag{48}$$

Now replacing $K_2(t)$ in (47) by $K_2(f)$, we get the generalization of $K_n(f)$ for all time-independent $K_n(-1)$:

$$K_n(f) = -\frac{2}{n}[K_2(f_1),\ K_{n-1}] \qquad f = \dot{f}_1 \quad K_{-1}(f) = 0 \quad n = 1, 2, \ldots\,. \tag{49}$$

Similarly to the KP case, in (49), we have defined the Lie algebraic meaning of D_y^{-1} by

$$D_y^{-1}C = yC + h(x, t) \tag{50}$$

where C is a constant and h is an arbitrary function of x and t which should be determined directly from the symmetry definition equations (8) and (9).

Because the various integral functions which should be fixed have been included in every high-order symmetry defined by (49), to get the concrete form of $K_n(f)$ is still quite difficult. In order to get the concrete form of $K_n(f)$, we return to (12), (13), (24) and (25) and fix the function $g(x)$.

From (8) and (9), we know that dependence of f (and its derivatives) for $K_n(f)$ must be linear because (8) and (9) are linear in U and H, (49) is linear in $f(= \dot{f}_1)$ and f is an arbitrary function of t. Accordingly, we know that $K_n(f)$ determined by (49) can also only have the forms (12) and (13) with (24). Now we determine $g(x)$ for $K_n(f)$ $(n = 0, 1, 2, \ldots)$ in a simple way using the same method as the KP case. If we say that x and y have a common dimension $[L]$, then from the IDLWE (5) and (6), we know that t, u and η should have the dimensions $[L]^2$, $[L]^{-1}$ and $[L]^{-2}$ respectively. Then from (42)–(44) and (49), we see that $U_0(f)/f$, $U_1(f)/f$, $U_2(f)/f, \ldots, U_n(f)/f$ have the dimensions $[L]^{-2}$, $[L]^{-3}$, $[L]^{-4}, \ldots,$ $[L]^{-n-2}$. That is to say, $f^{(n+1)}g_n(x)/f$, one term of $U_n(f)/f$, must have dimension $[L]^{-n-2}$, or equivalently, $g(x)$ must have dimension $[L]^n$. Accordingly, the only possible form of $g_n(x)$ for $K_n(f)$ given by (49) is

$$g_n(x) = \frac{x^n}{2^n n!} \qquad n = 0, 1, 2, \ldots \tag{51}$$

where the constant factor $1/2^n n!$ (which is not very important because any symmetry can be multiplied by an arbitrary constant) has been inserted such that the expressions of $K_n(f)$, $(n = 0, 1, 2, \ldots)$ obtained by (12), (13) and (24) with (51) and those defined in (49) coincide. Furthermore, if we want to find out all the possible symmetries given by (12) and (13) with (24) in addition to those given in (49), an arbitrary function in (24), $g(x)$, may be used. However, if we require the symmetries to be analytical at $x = 0$, then it is enough to find out all the possible independent symmetries shown by (12), (13) and (24) with $g(x)$ being given by (51) because an arbitrary analytical function can be expanded as a Taylor series. Nevertheless, if we do not require a symmetry to be analytical at $x = 0$, we should also discuss the symmetries with $g(x) = x^{-n}$, $n = 0, 1, 2, \ldots$. In this case, the symmetries (12) and (13) which have finite terms should be replaced by some infinite series expressions. We will treat such types of formal series symmetries in future studies, but not here. Finally, the generalized symmetries $K_n(f)$ can be written as

$$K_n(f) = \begin{pmatrix} (1/2^n n!)x^n + \sum_{k=1}^{n+1} f^{n-k+1} U_n[k] \\ \sum_{k=0}^{n} f^{n-k} H_n[k] \end{pmatrix} \qquad n = 0, 1, 2, \ldots \quad (52)$$

with

$$\begin{pmatrix} U_n[k] \\ H_n[k-1] \end{pmatrix} = -\frac{1}{2^n n!} \begin{pmatrix} -D_t - D_x u & -D_y^{-1} D_x^2 \\ -D_x \eta - D_x^2 D_y & -D_t - D_x u \end{pmatrix}^{k-1} \begin{pmatrix} (u x^n)_x \\ (\eta x^n)_x \end{pmatrix}$$
$$k = 1, 2, \ldots, n+1 . \quad (53)$$

After finishing the detailed calculations, we can prove that the generalized symmetries $K_n(f)$ also constitute a closed infinite-dimensional Lie algebra which is isomorphic to that of the KP equation [4]

$$[K_n(f_1),\ K_m(f_2)] = \tfrac{1}{2} K_{m+n-1} \left((m+1)\dot{f}_1 f_2 - (n+1)\dot{f}_2 f_1 \right) . \quad (54)$$

Here we would like to discuss some special cases of (54), instead of giving concrete verification.

(1) $f_1 = f_2 = \text{constant} = -1$. In this special case, we reobtain commuting algebra constituted by the time-independent symmetries

$$K_n(-1) = \frac{1}{2^n n!} \begin{pmatrix} -D_t - D_x u & -D_y^{-1} D_x^2 \\ -D_x \eta - D_x^2 D_y & -D_t - D_x u \end{pmatrix}^{n} \begin{pmatrix} -(u x^n)_x \\ -(\eta x^n)_x \end{pmatrix} \qquad n = 0, 1, 2, \ldots . \quad (55)$$

(2) $f = t$. In this case, $K_n(t) \equiv \tau_n$ can be called as the 'τ symmetry' that depend explicitly on the variables x, y and t linearly. This set of symmetries constitute a centreless Virasoro algebra (more precisely, the loop algebra of meromophic vector fields on circle):

$$[K_n(t),\ K_m(t)] = [\tau_n,\ \tau_m] = \tfrac{1}{2}(m-n) K_{m+n-1}(t) \qquad n,\ m \geqslant 0,\ K_{-n}(t) = 0 . \quad (56)$$

(3) $f = t^r$, $(r = 1, 2, \ldots)$. It is interesting that, in this special case, the general algebra (54) reduces to the algebra isomorphic to the w_∞ algebra

$$[K_n(t^r),\ K_m(t^s)] = \tfrac{1}{2} K_{m+n-1} \left((r(m+1) - s(n+1)) t^{r+s-2} \right) \quad (57)$$

which are widely used in other fields of physics [5]. So we call the algebra (54) the generalized w_∞ algebra.

Starting from the general symmetry expression (52) we can obtain not only the explicit expressions of the commuting symmetries (55), but also the explicit expressions of the time-dependent master-symmetries of degree k:

$$\tau_{k,n} = \frac{1}{2^n n!} \begin{pmatrix} -D_t - D_x u & -D_y^{-1} D_x^2 \\ -D_x \eta - D_x^2 D_y & -D_t - D_x u \end{pmatrix}^{n-k} \begin{pmatrix} -(u x^n)_x \\ -(\eta x^n)_x \end{pmatrix} \qquad k = 1, 2, \ldots, n . \quad (58)$$

3. Existence of another set of symmetries with arbitrary functions y

In addition to the $K_n(f)$ symmetries, there may be other types of symmetries for the IDLWE. For instance, we can look for the symmetries of the IDLWE which have the form

$$U(g) = \sum_{k=0}^{n} g^{(n-k)} U[k] \tag{59}$$

$$H(g) = \sum_{k=0}^{n+1} g^{(n+1-k)} H[k] \tag{60}$$

where g is an arbitrary function of y, $g^{(k)} = (\partial^k/\partial y^k)g(y)$ and $U[k]$ and $H[k]$ are y-independent explicitly. The explicit y dependencies of $U[k]$ and $H[k]$ have been separated out in $g^{(k)}$. Substituting (59) and (60) into (8) and (9) yields

$$U_t[0] = -H_{xx}[0] - (uU[0])_x \tag{61}$$

$$H_t[0] = -(uH[0])_x - U_{xx}[0] \tag{62}$$

$$U_t[k] + U_{ty}[k-1] = -(H_x[k] + u_y U[k-1] + uU[k] + uU_y[k-1])_x \qquad k = 1, 2, \ldots, n \tag{63}$$

$$H_t[k] = -(uH[k] + \eta U[k-1] + U_x[k] + U_{xy}[k-1])_x \qquad k = 1, 2, \ldots, n \tag{64}$$

$$U_{ty}[n] = -(H_x[n+1] + u_y U[n] + uU_y[n])_x \tag{65}$$

$$H_t[n+1] = -(uH[n+1] + \eta U[n] + U_{xy}[n])_x\,. \tag{66}$$

To find all the possible solutions of (61)–(66) is quite difficult. Here we give a special simple example. One can easily verify that

$$U[0] = -u_y \qquad H[0] = -\eta \qquad H[1] = -\eta_y \tag{67}$$

$$U[k] = H[k+1] = 0 \qquad k > 0 \tag{68}$$

i.e.

$$Y_0 = \begin{pmatrix} -g(y)u_y \\ -g(y)\eta_y - \dot{g}(y)\eta \end{pmatrix} \tag{69}$$

is a non-trivial solution of (61)–(66). The commutation relations of the algebra constituted by $K_n(f)$ and $Y(g)$ are given by (54) and

$$[Y_0(g),\ K_n(f)] = 0 \qquad n = 0, 1, 2, \ldots \tag{70}$$

$$[Y_0(g_1),\ Y_0(g_2)] = Y_0(\dot{g}_1 g_2 - \dot{g}_2 g_1)\,. \tag{71}$$

It is also interesting that there exists an infinite-dimensional Kac–Moody–Virasoro-type subalgebra of (54), (70) and (71),

$$X(h) = K_0(-h) \qquad T(f) = K_1(-f) \qquad Y(g) = Y_0(-g) \tag{72}$$

where h and f are two arbitrary functions of t and g is an arbitrary function of y; then we get an infinite-dimensional subalgebra of the IDLWE from (54), (70) and (71). The non-zero commutation relations of this subalgebra read

$$[T(f_1),\ T(f_2)] = T(f_1\dot{f}_2 - f_2\dot{f}_1) \tag{73}$$

$$[T(f),\ X(h)] = X(f\dot{h} - \tfrac{1}{2}h\dot{f}) \tag{74}$$

$$[Y(g_1),\ Y(g_2)] = Y(\dot{g}_2 g_1 - \dot{g}_1 g_2). \tag{75}$$

Using a known algorithm [15] and a MACSYMA program, this subalgebra was firstly found by Paquin and Winternitz [9] where $X(h)$, $T(f)$ and $Y(g)$ were expressed as the following totally equivalent forms:

$$X(h) = h(t)\partial_x + \dot{h}\partial_u \tag{76}$$

$$T(f) = f(t)\partial_t + \tfrac{1}{2}\dot{f}x\partial_x + \tfrac{1}{2}(\ddot{f}x - \dot{f}u)\partial_u - \tfrac{1}{2}\dot{f}(\eta_1 + 1)\partial\eta_1 \tag{77}$$

$$Y(g) = g(y)\partial_y - \dot{g}(\eta_1 + 1)\partial\eta_1 \qquad \eta_1 + 1 = \eta \tag{78}$$

and the corresponding Lie product is changed as $[A,\ B] = AB - BA$. Up to now, we have not yet found any other solutions of (61)–(66). Whether there exist any more non-trivial solutions of (61)–(66) is worthy of further study.

4. Summary and discussion

In this article, using the method of [4] given by the author for the KP equation, we get a set of generalized symmetries with an arbitrary function of t for the IDLWE in two space dimensions. The generalized infinite-dimensional Lie algebra constituted by this set of symmetries is a generalization of the well known w_∞ algebra which was firstly found for the KP hierarchy, Toda theory, string theory, two-dimensional gravity and membrane theory [5, 6]. Different from the KP equation, the generalized symmetry of the y-translation contains an arbitrary function of space (y) instead of time (t), which means there may exist another set of generalized symmetries with an arbitrary function of y. Unfortunately we do not yet have get any other non-trivial solutions except $Y_0(p)$. On the other hand, there may exist also some formal infinite series symmetries with arbitrary functions of t and negative powers of x which should be studied further.

There exist various other interesting problems worthy of further study. One of the most important problems may be how much and what type of (2+1)-dimensional integrable models possess such types of generalized symmetries and generalized w_∞ algebra in addition to the 2D IDLWE, KPE, 3D Toda field [16] and the Nizhnik–Novikov–Veselov equation [17]. Further study of the symmetries and generalized w_∞ algebra of IDLWE and the generalized symmetries and algebras for other (2+1)-dimensional models are in progress.

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References

[1] Kadomtsev B B and Petviashvili V I 1971 *Sov. Phy. Dokl.* **15** 539

[2] Cheng Y 1992 *Phys. Lett.* **166A** 217

[3] Lou S-Y, Ruan H-Y, Chen D-F and Chen W-Z *J. Phys. A: Math. Gen.* **24** 1455

[4] Lou S-Y 1993 Symmetries of the Kadomtsev–Petviashvili equation *J. Phys. A: Math. Gen.* **26** 4387

[5] Kogan I I 1992 *Mod. Phys. Lett.* A **7** 3717
Bakas I 1989 *Phys. Lett.* **228B** 57
Morozov A 1991 *Nucl. Phys.* B **357** 620
Pope C N, Romans L J and Shen X 1990 *Phys. Lett.* **236B** 173; 1990 *Nucl. Phys.* B **339** 191
Bergshoef E, Blencowe M P and Stelle K S 1990 *Commun. Math. Phys.* **128** 128
Bakas I, Khesin B and Kiritris E 1993 *Commun. Math. Phys.* **151** 233

[6] Das A, Sezgin E and Sin S J 1991 *Preprint* IC/91/383; Sezgin E 1991 *Preprint* IC/91/206, CTP TAMU-9/91; Awada M A and Sin S J 1990 *Preprint* UFIFT-HEP-90-30

[7] David D, Kamran N, Levi D and Winternitz P 1986 *J Math. Phys* **27** 1225

[8] Boiti M, Leon J J P and Pempinelli F 1987 *Inverse Problems* **3** 371

[9] Paquin G and Winternitz P 1990 *Physica* **46D** 122

[10] Lou S-Y 1994 *Math. Meth. Appl. Sci.* in press

[11] Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* **30** 2201
Lou S-Y 1990 *Phys. Lett.* **151A** 133
Clarkson P A 1989 *J. Phys. A: Math. Gen.* **22** 2355, 3821
Clarkson P A 1992 *Nonlinearity* **5** 453
Lou S-Y 1990 *J. Phys. A: Math. Gen.* **23** L649
Lou S-Y and Ni G-J 1991 *Commun. Theor. Phys.* **15** 465
Lou S-Y, Ruan H-Y, Chen D-F and Chen W-Z 1991 *J. Phys. A: Math. Gen.* **24** 1455
Lou S-Y 1992 *J. Math. Phys.* **33** 4300

[12] Lou S-Y 1993 *Phys. Lett.* **176A** 96

[13] See for example Gu C-H, Guo B-L, Li Y-S, Cao C-W, Tian C, Tu G-Z, Hu H-S, Guo B-Y and Ge M-L 1990 *Soliton Theory and its Application* (Hangzhou: Zhejiang) pp 216–67

[14] Fuchssteiner B 1983 *Prog. Theor. Phys.* **70** 1508

[15] Olver P J 1986 *Applications of Lie Group to Differential Equations* (Berlin: Springer)

[16] Lou S-Y 1993 Symmetries and algebras of integrable models *Int. Conf. on Interface between Physics and Mathematics (Hangzhou, 1993)*; 1993 Generalized symmetries and w_∞ algebras in 3D Toda field theory *Phys. Rev. Lett.* **71** 4099

[17] Lou S-Y 1994 Symmetry algebras of the potential Nizhnik–Novikov–Veselov model *J. Math. Phys.* **35** in press